

A new upper bound for the clique cover number with applications ^{*}

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Abstract

Let $\alpha(G)$ and $\beta(G)$, denote the size of a largest independent set and the clique cover number of an undirected graph G . Let H be an interval graph with $V(G) = V(H)$ and $E(G) \subseteq E(H)$, and let $\phi(G, H)$ denote the maximum of $\frac{\beta(G[W])}{\alpha(G[W])}$ overall induced subgraphs $G[W]$ of G that are cliques in H . The main result of this paper is to prove that for any graph G

$$\beta(G) \leq 2\alpha(H)\phi(G, H)(\log \alpha(H) + 1),$$

where, $\alpha(H)$ is the size of a largest independent set in H . We further provide a generalization that significantly unifies or improves some past algorithmic and structural results concerning the clique cover number for some well known intersection graphs.

1 Introduction

Throughout this paper $G = (V(G), E(G))$ is a simple undirected graph with $|V(G)| = n$. Let H be a subgraph of G . We denote by $\beta(H)$ the clique cover number of G . We further denote by $\alpha(H)$ and $\omega(H)$ the cardinality of a largest independent set and a largest clique in H , respectively. Let $G[W]$ denote the induced graph in G on the vertex set $W \subseteq V$.

Clearly $\beta(G) \geq \alpha(G)$, for any G . In addition, there exist graphs G with arbitrary large $\beta(G)$ so that $\alpha(G) \leq 2$ [14]. Consequently, given a graph G , one can not expect to have an upper bound for $\beta(G)$ that involves $\alpha(G)$ only. Furthermore, the problems of computing $\beta(G)$ exactly for a graph G , or even approximating it, are known to be computationally hard [12], [2]. Hence, deriving any general upper bound for the clique cover number that may offer algorithmic consequences is a significant move forward.

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Throughout this paper we say a graph H is a *supergraph* of G , if $V(G) = V(H)$ and $E(G) \subseteq E(H)$. Let H be a supergraph of G . Define $\phi(G, H)$ as

$$\phi(G, H) = \max_{\substack{W \subseteq V \\ H[W] \text{ is a clique}}} \left\{ \frac{\beta(G[W])}{\alpha(G[W])} \right\}.$$

An interval graph is the intersection graph of a set of intervals on the real line [16],[4]. The main contribution this paper is to prove the following.

Theorem 1.1 *Let H be an interval supergraph of G , then*

$$\beta(G) \leq 2\alpha(G)\phi(G, H)(\log(\alpha(H)) + 1).$$

Note that $\alpha(H) \leq \alpha(G)$ and hence Theorem 1.1 implies that $\beta(G) \leq 2\alpha(G)\phi(G, H)(\log(\alpha(G)) + 1)$, for any graph G .

We further generalize Theorem 1.1 to the case $E(G) = \cap_{i=1}^t E(H_i)$, where the supergraph $H_i, i = 1, 2, \dots, t-1$, is an interval graph and the supergraph H_t is a perfect graph. This generalization significantly unifies or improves some past algorithmic and structural results concerning the clique cover and the maximum independent set problems for some well known intersection graphs.

Specific applications include generalizing the results on the traversal number of rectangles [11], [3], [13], drastically improving an upper bound for the clique cover number of interval filament graphs[5] and improving the approximation factor for the polynomial time approximation of the clique cover number of interval filament graphs [9].

The strength of the work presented here is the generality that leads to the significant unification of several past results. In addition, the underlying method utilizes simple properties associated with the linear orderings and the clique separation of interval graphs, and thus may be of an independent interest.

2 Main Result

Two disjoint subsets $A, B \subseteq V(G)$ are *separated* in G , if there are no edges $ab \in E(G)$ with $a \in A$ and $b \in B$. Let $\pi : v_1, v_2, \dots, v_n$ be linear ordering of $V(G)$. For $i = 1, 2, \dots, n$ the sets $\{v_1, v_2, \dots, v_i\}$ and $\{v_i, v_{i+1}, \dots, v_n\}$ are denoted by V_i and V^i , respectively. We define $V_0 = \emptyset$. For $i = 1, 2, \dots, n$, let $N(v_i)$ denote the set of all vertices adjacent to v_i in V_{i-1} .

Let G be an interval graph whose interval representation is $J = \{[a_i, b_i] | i = 1, 2, \dots, n\}$. Thus G is the intersection graph of intervals in J . Let π be a linear ordering of elements in J in the increasing order of a_i , for $i = 1, 2, \dots, n$.

Then, π is called a *canonical linear ordering*. The following Lemma states some known properties concerning the canonical linear orderings of interval graphs [16], [4].

Lemma 2.1 *Let $\pi : v_1, v_2, \dots, v_n$ be a canonical linear ordering of vertices of an interval graph G . Then, for any $i = 1, 2, \dots, n$, $N(v_i)$ is a clique in G . In addition, for any $i = 1, 2, \dots, n$, $V_{i-1} - N(v_i)$ is separated from V^i in G .*

We now prove the main result.

Proof of Theorem 1.1 We will show that there is a clique cover $C(G)$, and an independent set $I(G)$ so that

$$|C(G)| \leq 2|I(G)|\phi(G, H)(\log(\alpha(H)) + 1).$$

To prove the claim, we induct on $\alpha(H)$. Our proof gives rise to a recursive divide and conquer algorithm for constructing $C(G)$ and $I(G)$. For $\alpha(H) = 1$, the claim is a direct consequence of the definitions, since in this case $\beta(H) = \alpha(H) = 1$. Thus H is a complete graph, and consequently $\beta(G) \leq \alpha(G)\phi(G, H)$. Now let $k \geq 2$, assume that the claim is valid for all graphs G and their interval supergraphs H with $\alpha(H) < k$, and let H be an interval supergraph of G with $\alpha(H) = k$. Let π be a canonical linear ordering, let $I, |I| = \alpha(H)$ be a maximum independent set in H , let j be the index of $\lfloor \frac{\alpha(H)}{2} \rfloor$ -th vertex in I in the ordering induced by π . Finally, let j^* be the index of the vertex in I that appears immediately after v_j , in the linear ordering induced by π . Let $V'_j = V_j - N(v_{j^*})$ and note that $H[V'_j]$ and $H[V^{j^*}]$ are interval graphs that contain independent sets of sizes $\lfloor \frac{\alpha(H)}{2} \rfloor$ and $\alpha(H) - \lfloor \frac{\alpha(H)}{2} \rfloor = \lceil \frac{\alpha(H)}{2} \rceil$, respectively. It follows that $\alpha(H[V'_j]) = \lfloor \frac{\alpha(H)}{2} \rfloor$, and $\alpha(H[V^{j^*}]) = \lceil \frac{\alpha(H)}{2} \rceil$, since by Lemma 2.1 V'_j is separated from V^{j^*} in H . Note further that $H[V'_j]$ and $H[V^{j^*}]$ are supergraphs of $G[V'_j]$ and $G[V^{j^*}]$, respectively, and hence V'_j and V^{j^*} are also separated in G . In addition, $\phi(G[V'_j], H[V'_j]) \leq \phi(G, H)$ and $\phi(G[V^{j^*}], H[V^{j^*}]) \leq \phi(G, H)$. Let $C(G[V'_j])$ and $I(G[V'_j])$ be clique cover and independent sets that are obtained by the application of the induction hypothesis to $G[V'_j]$ and $H[V'_j]$, then

$$|C(G[V'_j])| \leq 2|I(G[V'_j])|\phi(G, H)(\log \lfloor \frac{\alpha(H)}{2} \rfloor + 1).$$

Similarly, let $C(G[V^{j^*}])$ and $I(G[V^{j^*}])$ be the clique cover and the independent set obtained by applying the induction hypothesis to $G[V^{j^*}]$ and $H[V^{j^*}]$. Then,

$$|C(G[V^{j^*}])| \leq 2|I(G[V^{j^*}])|\phi(G, H)(\log \lceil \frac{\alpha(H)}{2} \rceil + 1).$$

Next, note that $H[N(v_{j^*})]$ is a clique in H , and thus, there is a clique cover $C(G[N(v_{j^*})])$ and an independent set $I(G[N(v_{j^*})])$ so that $|C(G[N(v_{j^*})])| \leq \phi(G, H)|I(G[N(v_{j^*})])|$. Now, let $C(G) = C(G[V_j^{j^*}]) \cup C(G[N(v_{j^*})]) \cup C(G[V_j'])$ and observe that $|C(G)| = |C(G[V_j'])| + |C(G[V_j^{j^*}])| + |C(G[N(v_{j^*})])|$, and hence by combining the last three inequalities, we obtain

$$|C(G)| \leq 2\phi(G, H)(|I(G[V_j'])| + |I(G[V_j^{j^*}])|)(\log \lceil \frac{\alpha(H)}{2} \rceil + 1) + \phi(G, H)|I(G[N(v_{j^*})])|.$$

Next define $I(G)$ to be the larger of the two independent sets $I(G[V_j^{j^*}]) \cup I(G[V_j'])$, $I(G[N(v_{j^*})])$. Then, the last inequality gives

$$|C(G)| \leq 2\phi(G, H)|I(G)|(\log \lceil \frac{\alpha(H)}{2} \rceil) + \frac{3}{2}.$$

To finish the proof, observe that $\log(\lceil \frac{i}{2} \rceil + \frac{3}{2}) \leq \log(i) + 1$ for any integer $i, i \geq 2$.

□

An immediate consequence of Theorem 1.1 is the following.

Corollary 2.1 *Let H_1 and H_2 be supergraphs of G , that are an interval graph and a perfect graph, respectively. If $E(G) = E(H_1) \cap E(H_2)$, then there is a clique cover $C(G)$ and an independent set $I(G)$, in G , so that*

$$|C(G)| \leq 2|I(G)|(\log(\alpha(H_1)) + 1).$$

Moreover, $I(G)$ and $C(G)$ can be computed in polynomial time.

Proof. Let $W \subseteq V(G)$ so that $H_1[W]$ is a clique. Then $G[W] = H_2[W]$, since $E(G) = E(H_1) \cap E(H_2)$. Thus $\beta(G[W]) = \alpha(G[W])$, since H_2 is a perfect graph. This implies that $\phi(G, H_1) = 1$, and the claims for the upper bound on $C(G)$ follow. To finish the proof, observe that for any subgraph of a perfect graph the clique cover and the independent set can be computed in polynomial time [8]. □

Remark. The time complexity of constructing $I(G)$ and $C(G)$ is $O(\hat{T}(n) + n^2 \log(n))$, where $\hat{T}(n)$ is the time that it takes to compute the clique cover number and the maximum independent set in any subgraph of the perfect graph H_2 . Note that $\hat{T}(n)$ is a polynomial of n , for the perfect graph H_2 . However, the degree of the polynomial depends on the structure of the H_2 . For instance, $\hat{T}(n) = O(n^2)$, when H_2 is an incomparability graph.

Using induction one can generalize Corollary 2.1.

Corollary 2.2 *Let $t \geq 2$, let H_1, H_2, \dots, H_{t-1} be interval super graphs of G , and let H_t be a perfect supergraph of G . If $E(G) = \cap_{i=1}^t E(H_i)$, then there is a vertex cover $C(G)$ and independent set $I(G)$, in G , so that*

$$|C(G)| \leq 2^{t-1} |I(G)| \prod_{i=1}^{t-1} (\log(\alpha(H_i)) + 1).$$

Moreover, $I(G)$ and $C(G)$ can be computed in polynomial time.

3 Applications

In this section we point out a few applications. Gavril [7] introduced the class of *interval filament* graphs which is an important class of intersection graphs and has shown that for any graph G in this class $E(G) = E(H_1) \cap E(H_2)$, where H_1 and H_2 are supergraphs of G which are an interval graph and an incomparability graph, respectively. This important class of graphs contains the classes of incomparability, polygon circle, chordal, circle, circular arc, and outer planar graphs. Gavril [7] has shown that $\alpha(G)$ and $\omega(G)$ can be computed in low-order polynomial time for any interval filament graph G . However, since the problem of computing $\beta(G)$ for a circle graph G is *NP-hard* [9], computing the clique cover number is also *NP-hard* for interval filament graphs.

Cameron and Hoang [5] have shown that for any interval filament graph G , $\beta(G) = O(\alpha(G)^2)$. One can also obtain this result by the application of the general method of Pach and Töröcsik [15]. The application of Corollary 2.1 to an interval filament graph G gives $\beta(G) = O(\alpha(G) \log \alpha(G))$, a significant improvement over the upper bound in [5]. Since our method in Corollary 2.1 constructs a suitable clique cover $C(G)$ in polynomial time, it also improves the best known approximation factor for the polynomial time approximation of $\beta(G)$ in [9] from $O(\log(n))$ to $O(\log(\alpha(G)))$. We remark that the $O(\log(n))$ approximation factor of Keil and Stewart in [9] has been derived for a larger class of graphs, namely the class of subtree filament graphs, that contains the class of interval filament graphs.

Kostochka and Kratochvil [10] have shown $\beta(G) = O(\alpha(G) \log(\alpha(G)))$ for any polygon circle graph G . Since any polygon circle graph is also an interval filament, Corollary 2.1 also implies the result in [10].

Let G be the intersection graph of a set S of axis parallel rectangles in the plane. Karolyi [11] was the first to prove that $\beta(G) = O(\log(\alpha(G))\alpha(G))$. (For a simpler proof see [6].) Later, Agarwal et al [1] studied the related problem of computing $\alpha(G)$ which is known to be *NP-hard*, and designed an $O(n \log(n))$ algorithm to construct an upper bound of $O(\alpha(G) \log(n))$. Different algorithmic variations of the method of Agarwal et al have been studied in [3], particularly in view of the applications of computing $\alpha(G)$ in map labeling and cartography. It is readily seen that $E(G) = E(H_1) \cap E(H_2)$, where H_1 and H_2 are interval supergraphs of G . Specifically, the

interval orders \prec^1 and \prec^2 associated with H_1 and H_2 define the separation properties of rectangles in S , in the horizontal and vertical directions. Since any interval graph is perfect, Corollary 2.1 applies and gives $\beta(G) = O(\alpha(G) \log(\alpha(G)))$. In addition, Corollary 2.2 gives $\beta(G) = O(\alpha(G) \log^t(\alpha(G)))$ and constructs a suitable clique cover and an independent set in polynomial time, which is the same as the best known result in [13], for the generalization of the problem to R^t , $t \geq 2$. A similar result follows from Corollary 2.2, when G is the intersection graph of a set S of convex polygons in the plane, where each $P \in S$ is obtained by the translation and magnification of a particular convex polygon with t corners, for a constant $t \geq 3$.

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